# THE STRUCTURE OF MAGNETOHYDRODYNAMIC SHOCK WAVES IN A GAS WITH ANISOTROPIC CONDUCTIVITY

(STRUKTURA MAGNITOGIDRODINAMICHESKOI UDARNOI VOLNY V GAZE S ANIZOTROPNOI PROVODIMOST'IU)

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We shall designate as a gas having anisotropic conductivity a completely ionized gas in the presence of an external magnetic field and sufficiently rarefied so that the electrons have spiral paths. In other words, we suppose that in the problem under consideration

$$\omega\tau \gtrsim 1 \qquad \left(\omega = \frac{eH}{m_e c}\right) \tag{1}$$

Here $\omega$ is the Larmor frequency, and $\tau$ the time between collisions of the electron and ion.

Under conditions (1) for a completely ionized gas, the relation analogous to Ohm's law takes the form [1]

$$\sigma\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H} + \frac{1}{ne}\operatorname{grad} p_e\right) = \mathbf{j} + \frac{\omega\tau}{H}\mathbf{j} \times \mathbf{H} \tag{2}$$

where $\sigma$ is the conductivity of the medium in the absence of a magnetic field, $p_e$ the electron pressure, $n$ the number of electrons per unit volume, and $e$ the charge on the electron.

Furthermore, we assume that the characteristic speeds and linear dimensions of the problem are such that a hydromechanical description of the medium is permissible.

With these assumptions as to the properties of the gas, we consider the problem of the structure of a magnetohydrodynamic shock wave. This problem was set forth in [2], but the solutions obtained there are exact solutions of only part of the equations describing the problem of shock-wave structure, and may therefore lead to qualitative conclusions that do not apply to the full system of equations.

A shock wave is a surface of discontinuity separating two moving streams of ideal gas of which the thermodynamic parameters, and also the intensity of the electromagnetic field, are connected by known relations which express the laws of conservation of mass, momentum, energy, normal component of magnetic field, and tangential component of electric field.

In considering the problem of shock-wave structure, it is customary to suppose that the shock wave appears as a narrow zone of change of the flow parameters, in which dissipation plays an essential role. Outside this zone the flow parameters change slowly, taking values at infinity that satisfy the conservation laws. In a system of coordinates in which the shock wave is at rest, the flow is a steady one-dimensional flow of non-ideal gas.

We will assume that only dissipation of energy of the electric current plays an essential role within the shock-wave zone. We shall neglect dissipation of energy as a result of viscosity and heat conduction.

The system of equations describing one-dimensional steady motion of a perfect gas under the assumptions adopted results in five algebraic relations expressing the laws of conservation of mass, three components of momentum, and energy, and to two differential equations obtained from (2) and describing the variation of magnetic field within the wave:

$$\rho u = m$$

$$mu + p + \frac{H^2}{8\pi} = J_1, \qquad mv - \frac{H_x}{4\pi} H_y = J_2, \qquad mv - \frac{H_x}{4\pi} H_z = J_3 \tag{3}$$

$$m\left(\frac{u^2 + v^2 + w^2}{2} + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right) = \varepsilon$$

$$\frac{dH_z}{dx} = \beta\left[\alpha H_x (uH_y - vH_x) - (wH_x - uH_z)\right]$$

$$\frac{dH_y}{dx} = \beta\left[\alpha H_x (wH_x - uH_z) + (uH_y - H_x v)\right] \tag{4}$$

$$\alpha = \frac{\omega\tau}{H}, \qquad \beta = \frac{4\pi}{c^2}\frac{\sigma}{1 + \alpha^2 H_x^2}, \qquad H_x = \text{const}$$

The third scalar equation obtained from (2) may be considered as an equation for the determination of $E_x$ if the gradient of electron pressure is given.

We suppose the $x$-axis to be directed normal to the surface of the wave, and the $y$- and $z$-axes to lie in its plane. The system of coordinates is chosen so that ahead of the shock wave (that is, at $x = -\infty$) the magnetic field and velocity will be parallel; then $\mathbf{E} = 0$ at $x = -\infty$, and consequently $\mathbf{E}_\tau = 0$ by virtue of Maxwell's equations.

We determine the constants in Equations (3) from conditions ahead of the shock wave (at $x = -\infty$). The constants in Equations (4) are determined by the physical properties of the medium and the intensity of the magnetic field.

The solution of the problem of the structure of a shock wave is that solution of the system (3), (4) which assumes at $x = \pm\infty$ given finite values of the flow parameters, where these values for $x = -\infty$ and for $x = +\infty$ are connected by the conditions at a shock wave in an ideal gas. We note that the relations at a shock wave in the present case coincide with the usual relations for a magnetohydrodynamic shock wave.

We transform the relations (3) and (4) to dimensionless variables, relating all variables to the parameters of the oncoming stream:

$$u = u_0 u^*, \quad v = u_0 v^*, \quad w = u_0 w^*, \quad RT = \theta u_0^2, \quad u^* p = \rho_0 u_0^2 \theta$$
$$H_i = \sqrt{8\pi\rho_0 u_0^2} h_i \qquad (i = x, y, z) \tag{5}$$

Here $\theta$ is the dimensionless temperature, $h_i$ the dimensionless components of the vector of magnetic field intensity, $u^*$, $v^*$ and $w^*$ the dimensionless velocity components.

The choice of coordinate system (rotation about the $x$-axis) may always be made so that ahead of the shock wave

$$w_0 = H_{z0} = J_3 = 0 \tag{6}$$

With relations (6) it follows, from the fourth equation (3) and the fact that the vectors **U** and **H** remain parallel behind the wave and lie in a plane with the corresponding vectors ahead of the wave [3], that

$$w_1 = H_{z1} = 0 \tag{7}$$

We will denote by index 1 parameters behind the wave. In the notation (5) and under the conditions (6), Equations (3) and (4) may be put in the following form:

$$\rho_0 = u^* \rho$$
$$u^{*2} + \theta + u^*(h_y^2 + h_z^2) = u^*(J_1 - h_x^2) = J_1^* u^*$$
$$v^* - 2h_x h_y = J_2^*, \qquad w^* - 2h_x h_z = 0 \tag{8}$$
$$\frac{u^{*2} + v^{*2} + w^{*2}}{2} + \frac{\gamma}{\gamma - 1}\theta = \varepsilon^*$$

$$\frac{1}{\beta^*} h_z' = (u^* - 2h_x^2)(\alpha^* h_x h_y + h_z) - \alpha^* h_x^2 J_2^*$$

$$\frac{1}{\beta^*} h_y' = (u^* - 2h_x^2)(h_y - \alpha^* h_x h_z) - h_x J_2^* \tag{9}$$

$$x^* = \frac{\sigma x}{c}, \qquad J_2^* = \frac{h_{y0}}{h_x}(1 - 2h_x^2), \qquad J_1^* = 1 + \theta_0 + h_{y0}^2$$

$$\varepsilon^* = \frac{1}{2}\left(1 + \frac{h_{y0}^2}{h_x^2}\right) + \frac{\gamma}{\gamma - 1}\theta_0, \quad \alpha^* = \sqrt{8\pi\rho_0 u_0^2}\,\alpha, \quad \beta^* = \beta\frac{cu_0}{\sigma}$$

The prime indicates the derivative with respect to the dimensionless coordinate $x^*$.

Eliminating $v^*$, $w^*$ and $\theta$ from relations (8), we obtain one relation connecting $h^*$, $h_y$ and $h_z$

$$\tfrac{1}{2}(\gamma + 1) u^{*2} + \gamma u^* (h_y^2 + h_z^2) - 2(\gamma - 1) h_x^2 (h_y^2 + h_z^2) -$$
$$- \gamma J_1^* - 2(\gamma - 1) J_2^* h_x h_y - \tfrac{1}{2}(\gamma - 1) J_2^* + (\gamma - 1)\varepsilon^* = 0 \tag{10}$$

The relation (10) describes in the space of $u^*$, $h_y$, $h_z$ a certain surface upon which must lie the integral curve of the system of equations (9) describing the structure of the shock wave. It is easy to verify that the initial point ($u^* = 1$, $h_y = h_{y0}$, $h_z = 0$) belongs to the surface (10). The section of the surface (10) by the plane $h_z = 0$ represents the curve corresponding to the problem of the structure of a magnetohydrodynamic shock wave in a gas with isotropic conductivity, if only one coefficient of dissipation $\sigma$ is considered:

$$\tfrac{1}{2}(\gamma + 1) u^{*2} + \gamma u^* h_y^2 - 2(\gamma - 1) h_x^2 h_y^2 - \gamma J_1^* u^* -$$
$$- 2(\gamma - 1) J_2^* h_x h_y - \tfrac{1}{2}(\gamma - 1) J_2^{*2} + (\gamma - 1)\varepsilon^* = 0 \tag{11}$$

This curve was investigated in detail in [ 4 ].

The surface (10) consists of circles lying in planes perpendicular to the $u^*$-axis, whose centers are on the hyperbola

$$h_y = \frac{(\gamma - 1) J_2^*}{\gamma u^* - 2(\gamma - 1) h_x^2} \tag{12}$$

One of the possible forms of the surface (10) is shown in Fig. 1.

At the points of the integral curve of the system (9) that correspond to $x = \pm\infty$, the right-hand sides of Equations (9) should vanish; that is, at $x = \pm\infty$

$$(u^* - 2h_x^2)(\alpha^* h_x h_y + h_z) - \alpha^* h_x^2 J_2^* = 0 \tag{13}$$

$$(u^* - 2h_x^2)(h_y - \alpha^* h_x h_z) - h_x J_2^* = 0 \tag{14}$$

Hence it follows that in the space of $u^*$, $h_y$, $h_z$ points corresponding to conditions ahead of and behind the shock wave should lie on the intersections of the surface (10) and the hyperbolic cylinders (13) and (14).

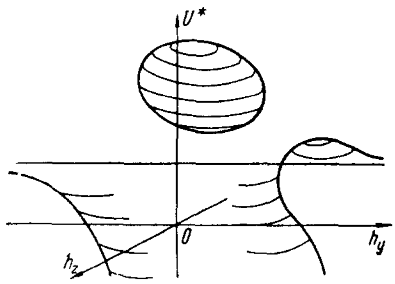

Fig. 1.

It is easy to verify that the cylinders (13) and (14) intersect in the hyperbola

$$(u^* - 2h_x^2) h_y - h_x J_2^* = 0 \qquad (15)$$

which lies in the plane $h_z = 0$.

The generators of the cylinders (13) and (14) are perpendicular to each other.

Thus points corresponding to conditions behind and ahead of the shock wave lie in the plane $h_z = 0$ (which follows also from (7)), and are points of intersection of the curve (11) with the hyperbola (15). In [4] and [5] it was shown that there are no more than four such points, where if the stability of the waves is limited in the sense of [6] it is necessary to consider only two of these points of intersection, one of which will correspond to conditions ahead of the shock wave ($u^* = 1$, $h_y = h_{y0}$), and the other to conditions behind the shock wave. For definiteness we will assume that everywhere $J_2^* > 0$. Then fast waves (waves with intensification of the field), for which $2h_x^2 < 1$, will correspond to transitions $0 < h_{y0} < h_{y1}$, and slow waves (waves with weakening of the field) will correspond to transitions $h_{y0} < h_{y1} < 0$.

The angle of inclination of the tangent to the curve (11) at the point ($u^* = 1$, $h_y = h_{y0}$) is

$$\frac{du^*}{dh_y} = -\frac{2h_{y0}}{1 - \gamma\theta_0} \qquad (16)$$

and the angle of inclination of the tangent to the hyperbola (15) at this same point is

$$\frac{du^*}{dh_y} = -\frac{1 - 2h_x^2}{h_{y0}} \qquad (17)$$

If the right-hand sides are equal, that is, the curves (11) and (15) are tangent, the corresponding speed of the stream is equal to one of the magneto-sonic speeds.

Depending upon the parameters behind and ahead of the shock wave [4], various types of intersection of the curves (11) and (15) are possible, as shown in Fig. 2. (Cases *1* and *2* correspond to fast waves and *3*, *4* and

5 to slow waves. In case 5 the curves do not join).

For investigation of the local properties of the point of intersection of the curves (11) and (15), we may suppose, without loss of generality, that at this point $u^* = 1$ and $h_y = h_{y0}$. This can be achieved by changing the scale along the coordinate axes; that is, by referring all quantities in (15) to the values of the parameters at the point under consideration.

At points *1A, 1B, 2A, 4A, 5A* the speed of the stream is greater than the usual speed of sound and

$$1 - \gamma\theta_0 = 1 - \frac{1}{M_0^2} > 0 \qquad (18)$$

At the remaining points

$$1 - \gamma\theta_0 < 0 \qquad (19)$$

From geometric considerations it is evident that at points *1A, 2A, 3B, 4B, 5B*

$$\frac{2h_{y0}}{1 - \gamma\theta_0} < \frac{1 - 2h_x^2}{h_{y0}} \qquad (20)$$

and at points *1B, 3A*

$$\frac{2h_{y0}}{1 - \gamma\theta_0} > \frac{1 - 2h_x^2}{h_{y0}} \qquad (21)$$

The solution of the problem of shock-wave structure is the integral curve of the system (9) passing through the points of intersection of the curves (11) and (15) and lying on the surface (10).

If $u^*$ is eliminated from Equation (9) by means of (10), the system of two equations (9) is equivalent to the one equation

$$\frac{dh_z}{dh_y} = \frac{(u^* - 2h_x^2)(\alpha^* h_x h_y + h_z) - \alpha^* h_x^2 J_2^*}{(u^* - 2h_x^2)(h_y - \alpha^* h_x h_z) - h_x J_2^*} \qquad (22)$$

where the points of intersection of the curves (11) and (15) are singular points of this equation. The qualitative behavior of the solution and, consequently, the qualitative

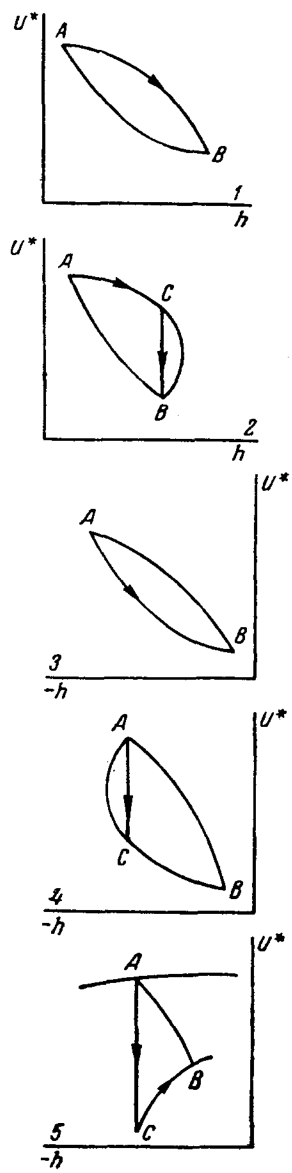

Fig. 2.

nature of the flow behind the zone representing a shock wave in a gas with anisotropic conductivity, will depend on the character of these singular points.

In order to ascertain the character of the singular points of Equation (22) it is necessary in the neighborhood of these points to replace the surface (10) by its tangent plane and proceed in the usual way. The character of the singular points is determined by the sign of the discriminant $D$ of the characteristic equation formed from the coefficients of the linear terms in the numerator and denominator of the right-hand side of Equation (22):

$$D = \left[-\frac{h_{y0}^2}{1-\gamma\theta_0} + (1-2h_x^2)\right]^2 + (1-2h_x^2)\left[\frac{2h_{y0}^2}{1-\gamma\theta_0} - (1-2h_x^2)\right] \times$$

$$\times (1+\alpha^{*2}h_x^2) = \frac{h_{y0}^4}{(1-\gamma\theta_0)^2} + \alpha^{*2}h_x^2(1-2h_x^2)\,h_{y0}\left[\frac{2h_{y0}}{1-\gamma\theta_0} - \frac{1-2h_x^2}{h_{y0}}\right] \quad (23)$$

The character of the singularity is a local property of the point, so that it is always possible to suppose that the singular point has coordinates $u^* = 1$, $h_y = h_{y0}$, $h_z = 0$ and distinguish the singular points according as the inequalities (18) to (21) are satisfied for a given point.

If $D > 0$, the singular point is a node if the inequality (21) is satisfied, and a saddle if the inequality (20) is satisfied. If $D < 0$, the singular point is a focus.

Hence it follows that points *1B* and *3A* are saddles. Points *1A*, *2A*, *3B*, *4B*, *5B* are nodes if $D > 0$ ($\alpha^{*2}h_x^2$ small), and foci if $D < 0$ ($\alpha^{*2}h_x^2$ large). At points *2B* and *5A* the square brackets in (23) are negative, and there is consequently a node if $\alpha^{*2}h_x^2$ is small, and a focus if $\alpha^{*2}h_x^2$ is large.

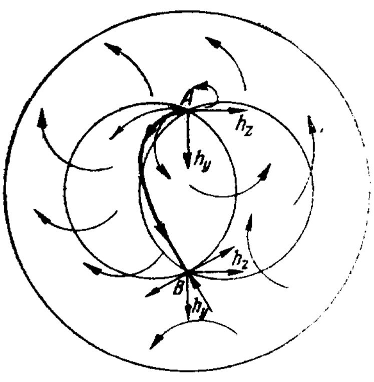


Fig. 3.

The direction of movement along the integral curve as $x^*$ increases at any point depends on the position of that point with respect to the cylinders (13) and (14). If the line of intersection of (13) and (14) with the surface (10) is projected onto the tangent plane to (10) at the singular point, then it is possible to draw in the tangent plane the field of isoclines and to determine the direction along the integral curves in the vicinity of each singular point.

Such an investigation shows that for fast waves all nodes and foci are outgoing (the integral curve leaves these points as $x$ increases), and for slow waves the integral curve always enters nodes and foci.

Hence it follows immediately, for example, that in cases *2*, *4* and *5* it is impossible to construct a continuous flow inside the shock-wave zone. In these cases, a gasdynamic discontinuity appears inside the region of the wave, which is possible in our formulation of the problem as a consequence of the neglect of viscosity and heat conduction.

In Figures 3 and 4, the general form of the integral curves is shown as an example, and the integral curves corresponding to the problem of shock-wave structure are drawn qualitatively for cases *1* and *2*. The figures give a view of the surface (10) in the direction of the $h_y$-axis.

The closed curves represent the projections of lines of intersection of the surface (10) with the cylinders (13) and (14). The figures correspond to small values of $a^{*2}h_x^{\ 2}$, when point $A$ is a node.

The foregoing analysis makes it possible to describe qualitatively the behavior of the solution within the zone representing the shock wave in all the cases *1* to *5*.

*1*. If conditions are such that the spiral path of the electrons is not large ($a^{*2}h_x^{\ 2}$ small), point $A$ is a node. Point $B$ is a saddle. The character of the integral curves is shown in Fig. 3.

In this case there exists a unique integral curve joining points $A$ and $B$ and describing the solution of the problem of shock-wave structure. The quantities $u^*$ and $h_y$ change monotonically through the wave.

The component $h_z$ of the magnetic field first grows from zero to a certain value, and then decreases to zero.

If the spiral path of the electrons is large, point $A$ is a focus. The end of the magnetic field intensity vector describes at first a certain spiral curve in space, and then decreases to zero.

In the general cases the motion within the wave is continuous.

*2*. In the vicinity of point $A$ the behavior of the solution is the same as in case *1*. But because in this case it is impossible to construct a continuous solution between points $A$ and $B$ (this is connected with the fact that in such a solution there would be a point where $M = 1$), a gasdynamic shock appears inside the shock-wave zone. The solution of the problem of shock-wave structure corresponds to an integral curve leaving $A$ that passes through a point $C(u^* = u_1^*,\ h_y = h_{y1},\ h_z = 0)$. Within the wave the flow parameters at first vary continuously (to point $C$), and there then arises a gasdynamic shock with continuous magnetic field that takes the gas from condition $C$ to condition $B$. The character of the integral curves for small $a^{*2}h_x^{\ 2}$ is shown in Fig. 4.

*3*. Point $A$ is a saddle; point $B$ a node or focus. The motion inside the wave-zone proceeds continuously according to an integral curve corresponding to one of the principal directions of the saddle $A$. The variation of magnetic field near point $B$ is analogous to its variation near point $A$ in case *1*.

*4*. Points $A$ and $B$ are nodes or foci into which integral curves enter. A continuous solution is impossible. The wave-zone begins with a gasdynamic shock at which the magnetic field is continuous, followed by a region of continuous variation of the parameters.

The character of the behavior of the magnetic field near point $B$ is analogous to that in case *3*.

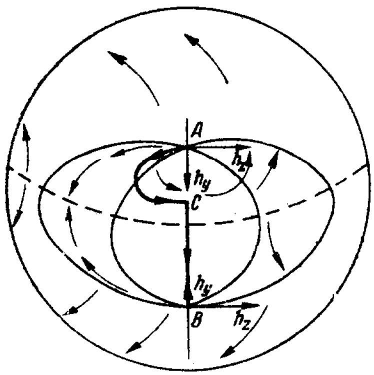

Fig. 4.

*5*. The character of the solution is the same as in case *4*.

Thus the character of the flow within the zone representing the shock wave depends essentially upon the magnitude of the spiral path of the electrons and upon the character of the wave itself (fast or slow). Independently of the form of the wave, the solution describing its structure corresponds to a three-dimensional flow.

The magnetic field intensity vector varies inside the wave in such a way that its end may, for large spiral paths of the electrons, rotate so as to describe a spiral trajectory, but this turning does not correspond to a rotation of the vector $\mathbf{H}$ about the $x$-axis. If the spiral path of the electrons tends toward zero ($a^* \to 0$), the shock wave is converted into an ordinary magnetohydrodynamic shock wave.

The thickness of the shock wave, if it is strong, may be taken as the distance in which the magnetic field and all other quantities change by an amount of the order of their values ahead of the shock wave. If the shock-wave thickness is determined in this way, it follows from (4) that if $\omega\tau \sim 1$, then

$$l \sim \frac{c^2}{4\pi\sigma u_0}(1 + \alpha^2 H_x^2) \sim \frac{c}{4\pi\sigma u_0}(1 + \omega^2\tau^2)$$

if $\omega\tau \gg 1$, then

$$l \sim \frac{c^2}{4\pi\sigma u_0}\frac{1 + \alpha^2 H_x^2}{\alpha H_x} \sim \frac{c^2}{4\pi\sigma u_0}\omega\tau$$

These relations show that the thickness of a magnetohydrodynamic shock wave is greater in a gas with anisotropic conductivity than the thickness of a conventional magnetohydrodynamic shock wave

$$l_0 \sim \frac{c^2}{4\pi\sigma u_0}$$

Furthermore, for large spiral path of the electrons ($\omega\tau \gg 1$) the thickness of the shock wave in a gas with anisotropic conductivity becomes of the order of magnitude of the Larmor radius of the ions

$$l \sim \frac{c^2}{4\pi\sigma u_0}\omega\tau = \frac{cH}{4\pi u_0 nl} \sim \frac{cH^2 u_0 m_i}{4\pi\rho u_0^2 lH} \sim R_i$$

in the case that the energy of the magnetic field is comparable with the kinetic energy of the medium ($H^2/8\pi \sim \rho u_0^2$).

Projecting (2) onto the $x$-axis, we obtain

$$-\frac{\omega\tau}{H}\frac{d}{dx}(H_y^2 + H_z^2) = \frac{\sigma}{c}(vH_z - wH_y) + \frac{1}{ne}\frac{dp_e}{dx} + \sigma E_x \qquad (24)$$

As mentioned earlier, this equation serves to determine $E_x(x)$ for given $p_e$ and solution of the system (4). If the chaotic speeds of the ions and electrons are equal in magnitude, and so are their numbers per unit volume, then it is possible to assume that $2p_e = p(p_i = p_e)$.

Then since $E_x(x)$ is found, the equation $dE_x/dx = 4\pi\rho_e$ determines the space-charge density inside the wave as a function of $x$.

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